Note on Water Measurements by Frontinus

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Abstract

Frontinus measures the amount of water delivered via various pipes or ducts simply by the area of the duct's cross-section, with the underlying assumption that the amount of water is proportional to that cross-area. It is in this way that he gauges the capacities of all his pipes. It must have led to serious mistakes, witness the numerous discrepancies that Frontinus discovers and blames on illegal water-use. The discrepancies must have been partly due to the fact that Frontinus generally ignores: the delivered amount of water also depends on the speed of the flow through the pipe. We now know, that this speed depends on the area of the pipe's cross-section in a way that is different from proportionality, and furthermore, it also depends on the shape of the cross-section, the length of the pipe, the roughness of its walls, and, perhaps most obviously to us, on the level of water in the reservoir from which the water is flowing. Frontinus demonstrates (§ 35, 70, 73) that he appreciated at least the fact of the dependence on pipe length and water level, but even so he disregards it in all his calculations, probably because he had no way of quantifying that dependence.

The development of fluid dynamics – the branch of physics concerned with the motion of fluids – only began in earnest in the 19th century, so Frontinus' vague ideas about it are quite pardonable. The naive nature of his system for measuring flowing water has been noted a long time ago (see, for example, [1-3]). Frontinus "measures" the amount of water flowing through a pipe or any other conduit by measuring the cross-sectional area of that conduit. To us, the flaw in his logic is manifest: one also needs to know the speed with which the water moves through the pipe. Then, the volume of the water Q delivered by the pipe per unit of time is given as the product of the cross-sectional area A and that speed:

$$Q = AV. \tag{1}$$

But how does one determine the speed of water flowing through a pipe? The works cited above concentrate on the fact that this speed depends on the difference of the water pressure between the two ends of the pipe. Indeed, a basic application of the conservation of energy produces the formula for the speed with which water flows out of a reservoir, historically known as Torricelli's law:

$$V = \sqrt{2gh}, \text{ hence } Q = A\sqrt{2gh}. \qquad (2)$$

Here V is the speed, $g \approx 980$ cm/sec^2 is the acceleration of gravity, and h is the height of the water level in the reservoir above the orifice ("head", in the parlance of hydraulic engineers). The pressure difference between the ends of the pipe is, in fact, proportional to the height h.

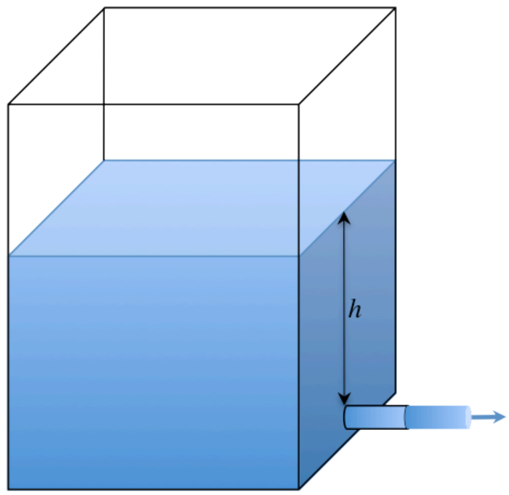

Figure 1. A model situation in which water is flowing out of a reservoir through a pipe. The height shown here determines the water pressure and is called "head".

Of course, it would not be correct to think that Frontinus was completely unaware of the importance of water speed. He talks about this factor in a few places (§ 35, 70, 73). Rather, it seems that he assumed it to be more or less a constant, apart from extreme cases. Indeed, if we assume that the "head", i.e. the height of water in the reservoir from which a pipe is drawing water, is constant, the speed according to Eq. 2 is constant, and so the flow rate Q is proportional to the cross-sectional area A, just as Frontinus would like it to be.

In reality, however, Frontinus' assumption is far from being the truth even if the "head" is constant, because Torricelli's law is true only in absence of friction and viscosity – an approximation reasonable only when the pipe is very short (i.e. essentially an orifice). Below I will attempt to explain the general effect of friction and viscosity. It is, of course, impossible to treat this subject rigorously without going into details that would be accessible only to a reader with some background in theoretical physics. I will try to give only the essence of the matter and refer sufficiently equipped readers to any of the existing textbooks on fluid dynamics, such as [4], for further details.

A fundamental fact about the flow of a fluid is that it can be in two radically different regimes: laminar and turbulent. In a laminar flow the fluid does not form eddies, and its velocity may be unchanging in time. A turbulent flow, on the contrary, is dominated by eddies. The analysis of the equations of fluid dynamics shows that the parameter, which determines whether a flow is laminar or turbulent, is the so-called Reynolds number, denoted Re and found approximately as

$$\mathrm{Re} = \frac{vD}{\eta}. \qquad (3)$$

Here v is the typical velocity of the fluid, D is the pipe's diameter, and η is the *viscosity* of the fluid. With a good precision, its value for water is $\eta = 0.01$ cm²/sec. The flow is laminar for lower Reynolds numbers, and turbulent for higher ones. For a flow in a round pipe the transition between the two regimes occurs somewhere in the vicinity of $\mathrm{Re} = 2{,}000$.

Laminar flow in a round pipe

A simple estimate will show that, in realistic circumstances, the flow in any of Frontinus' pipes would have a high Reynolds number, and so would have been turbulent. Nevertheless, the consideration of the laminar flow (which would take place, for instance, in a much narrower pipe) is instructive, because it introduces several useful concepts. In this case, the equations of fluid dynamics show that the velocity of the fluid is maximal at the axis of the pipe, and gradually reduces to zero at the walls.

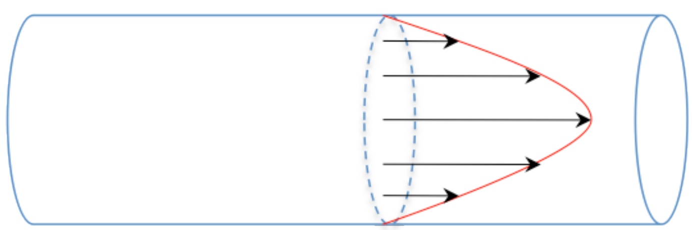

Figure 2. Schematic distribution of velocities in a laminar flow through a round pipe.

It turns out, that the profile of the velocities is actually a parabola. Thus, the fundamental fact is that there is no such thing as "the speed of water in a pipe": in reality, the velocity of the water varies from point to point in the cross-section of the pipe. Hence, V in Eq. 1-2 should be thought of only as an *average* speed. Another fundamental fact is that the elevation of the parabolic profile, i.e. the value of the velocity at the axis of the pipe, itself increases with the diameter of the pipe. Thus, *if we increase the diameter of the pipe, not only more water fits into it, but some of it will also move faster.* In other words, the average velocity V in Eq. 1 is not a constant, but itself depends on A, with the result that the flow rate grows not proportionally to A, but faster. Namely, in a laminar flow it turns out to be

$$Q = \frac{g}{8\eta\pi} \cdot \frac{h}{L} A^2, \quad (4)$$

where L is the length of the pipe. The length is assumed to be sufficiently large, in order for the laminar flow to settle into its pattern after entering the pipe. Thus, if the pipe in Fig. 1 is not assumed to be short, the flow rate is given by Eq. 4, not by Eq. 2. The difference is striking: instead of being proportional to the area, the flow rate grows as the square of that area, and depends on the pipe length, as well. The latter is easy to understand: the longer the walls, the more friction they exert on the water, slowing it down.

Let us stress again, that Eq. 4 is valid only as long as the flow is laminar, which will be the case as long as the cross-section A is small enough. If it were true for Roman pipes, the discrepancies between the reality and Frontinus' calculations would be truly enormous. For instance, he calculates the capacity of a *centenaria* to be about 81 *quinariae*, but in reality it would be about 6,640 *quinariae*! Luckily for him, the flow in his pipes was, as we mentioned above, turbulent, and in a turbulent flow the dependence on the cross-section is considerably attenuated compared with Eq. 4.

Turbulent flow in a round pipe

In case of turbulent flow, the equations of fluid dynamics cannot be solved exactly, so one has to enter the realm of approximations, semi-empirical results and substantially more complicated mathematics. One could not, for example, produce a static picture like Fig. 2 for turbulent flow, because local velocities of the fluid are constantly and chaotically changing in time, as the eddies are being produced. But if we wait for some time, and average the value and direction of the velocity in each observation point over that time, we will indeed find a picture similar to Fig. 2, only with a flatter profile.

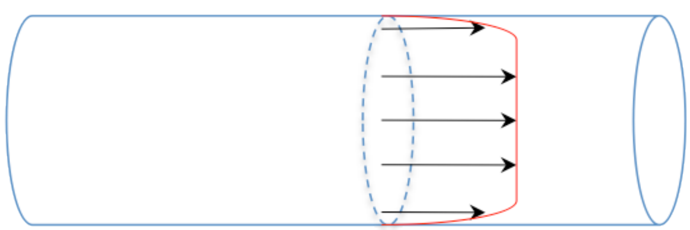

Figure 3. Schematic distribution of time-averaged velocities in a turbulent flow through a round pipe.

Now, the averaged velocities are almost the same throughout much of the cross-section, and then drop off in the region near the walls. A good approximation of the flow rate in this case is the following semi-empirical formula:

$$Q = A\sqrt{\frac{Dgh}{0.64L}} \ln\left(\frac{D}{2\eta}\sqrt{\frac{Dgh}{4L}}\right). \quad (5)$$

(Assuming that the pipe walls are not too rough; if they are, the formula will be different, but the modification is insignificant for our purposes). Despite its intimidating looks, this formula is not difficult to analyze at the level of detail that we are interested in. The logarithm is a very slowly changing function, so one only needs to concentrate on the expression preceding it. We observe that the additional dependence on the cross-sectional area comes via the diameter D under the square root. Hence, roughly speaking, this formula claims that the flow rate increases with the diameter proportionally to $D^2\sqrt{D}$, in other words, proportionally to $A\sqrt[4]{A}$.

This is, of course, a much milder deviation from simple proportionality of Eq. 2, than we saw in the laminar flow (Eq. 4). Nevertheless, the error quickly accumulates as one goes to larger pipes, and for a *centenaria* we would expect the capacity on the scale of 250 *quinariae*, instead of Frontinus' result of 81 *quinariae*. Finally, note that the flow rate of the turbulent flow also depends on the pipe length, although differently from the laminar case.

Conclusion

Frontinus ignores entirely the effect of the pipe length on the flow rate (even though he shows some awareness of it in § 35, this idea is not reflected in his calculations), and significantly misjudges the effect of the cross-sectional area of the pipe. This is in addition to his disregard of the changes in the water pressure, i.e. the "head" h, which was amply noticed in the literature before.

Yet another trap into which Frontinus fell is that he disregards the shape of the conduit, too. Note that the Equations 4 and 5 are written for a *round* pipe. In a conduit of a different shape (e.g. a square pipe, or even an open channel, such as he measured in § 65) the formulas would be yet again different. The rule of thumb is that for two conduits of the same cross-sectional area, the one with the smaller circumference will have greater capacity, since it is the contact between the fluid and the conduit walls that causes the flow to slow down. Hence, for example, a round pipe should deliver more water than a pipe with the cross-section of the same area but of a different shape, and an open channel may deliver more than a fully filled closed channel, because the top surface of the flowing water is free.

It comes, therefore, as no surprise, that Frontinus regularly finds enormous differences between the measured intakes and outputs of aqueducts, a fact, which he routinely blames on unauthorized water-users and dishonest aquarii.

References

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